Approximation of Boundary Control Problems on Curved Domains. I - The Neumann Case

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APPROXIMATION OF BOUNDARY CONTROL PROBLEMS ON CURVED DOMAINS. I - THE NEUMANN CASE*

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Abstract. In this paper we consider a Neumann control problem associated to a semilinear elliptic equation defined in a curved domain Ω. To deal with the numerical analysis of this problem, the approximation of Ω by an appropriate domain Ω_h (typically polygonal) is required. Then the same infinite dimensional control problem is formulated in Ω_h. We study the influence of the replacement of Ω by Ω_h on the solutions of the control problem. Our goal is to compare the optimal controls defined on $\Gamma = \partial\Omega$ with those defined on $\Gamma_h = \partial\Omega_h$ and to derive some error estimates. The use of a convenient parametrization of the boundary is needed for such estimates.

Key words. Neumann control, curved domains, error estimates, semilinear elliptic equations

AMS subject classifications. 49J20, 35J65

1. Introduction. In this paper we study the following optimal control problem

$$(\mathrm{P})\left\{\begin{array}{l}\min J(u) = \displaystyle\int_\Omega L(x, y_u(x))\,dx + \frac{N}{2}\int_\Gamma u^2(x)\,d\sigma(x)\\ \text{subject to } \ (y_u, u) \in (L^\infty(\Omega)\cap H^1(\Omega)) \times L^2(\Gamma),\\ \alpha \le u(x) \le \beta \quad \text{for a.e.} \quad x \in \Gamma,\end{array}\right.$$

where Γ is a smooth manifold, y_u is the state associated to the control u, given by a solution of the Neumann problem

$$\left\{\begin{array}{rcll} -\Delta y + a(x,y) &=& 0 & \text{in } \Omega,\\ \partial_\nu y &=& u & \text{on } \Gamma.\end{array}\right. \tag{1.1}$$

We precise later the assumptions about the data of the control problem (P).

The numerical computation of the solution of (P) requires the discretization of the state equation, typically by using finite elements. If Ω is a polygonal domain, then it is covered by the union of the triangles of the mesh and Γ remains invariable. Then problem (P) is approximated by a discrete problem (P_h) and it is possible to estimate the difference $\|\bar u - \bar u_h\|_{L^2(\Gamma)}$ between the different solutions of (P) and (P_h), see for instance [3] or [4]. In the problem that we are considering here, the situation is more complicated because the numerical analysis with finite elements requires the approximation of Ω by a new (typically polygonal) domain Ω_h, so that the comparison between the solutions $\bar u$ and $\bar u_h$ is more involved because $\bar u \in L^2(\Gamma)$ and $\bar u_h \in L^2(\Gamma_h)$, where Γ_h is the boundary of Ω_h. This difficulty can be overcome by using convenient parametrizations of Γ and Γ_h, but there are still some technical difficulties for the

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error analysis. In this paper we do not consider the numerical approximation of (P), we just analyze what happen if Ω is approximated by a polygonal domain Ω_h, and (P) is transformed in a new infinite dimensional control problem (P$_h$)

$$(\mathrm{P}_h)\left\{\begin{array}{l}\min J_h(u) = \displaystyle\int_{\Omega_h} L(x, y_{h,u}(x))\,dx + \frac{N}{2}\int_{\Gamma_h} u^2(x)\,d\sigma_h(x)\\ \text{subject to } \ (y_{h,u}, u) \in (L^\infty(\Omega_h)\cap H^1(\Omega_h))\times L^2(\Gamma_h),\\ \alpha \le u(x) \le \beta \quad \text{for a.e. } \ x \in \Gamma_h,\end{array}\right.$$

where $y_{h,u}$ is the solution of the problem

$$\left\{\begin{array}{rcll} -\Delta y + a(x,y) & = & 0 & \text{in } \Omega_h,\\ \partial_{\nu_h} y & = & u & \text{on } \Gamma_h.\end{array}\right. \tag{1.2}$$

In this paper we study the influence of a small change in the domain on the solutions of the control problem. In §6 we prove that the order of the approximation is $h^{5/3}$. This order has an interesting consequence. Indeed, to solve numerically a Neumann control problem, piecewise constant or piecewise linear functions are typically taken to approximate the controls. In both these cases, the maximal order of the error estimates is h or $h^{3/2}$, respectively; see [3]. A consequence of our estimate is that it is not worthy to consider any better approximation Ω_h of Ω than the polygonal one because it does not lead to any improvement in the order of the convergence of the numerical approximations. Even, if we follow the procedure suggested by Hinze in [8], where no control discretization is considered, just the state and adjoint states are discretized, we cannot improve the order of convergence by using a better approximation Ω_h of Ω, unless finite elements of order higher than one are considered to solve the state and adjoint state equations. However, in the last case the implementation is much more involved if we do not discretize the controls, the computational complication being increased by the fact that the control is the Neumann boundary condition.

The plan of the paper is as follows. In §2 we fix the notation, we introduce the assumptions, we study the existence, uniqueness and regularity of the state equation (1.1) as well as the existence of a solution for problem (P). In section §3 the first and second order optimality conditions for (P) are established, which are the essential tool to derive the error estimates. The domains Ω_h, $h > 0$, are introduced in §4. Beside that, in §4 we define a one-to-one mapping $g_h : \Gamma_h \longrightarrow \Gamma$ that allows us to compare the solutions $\bar u$ of (P) and $\bar u_h$ of (P$_h$) in the norm $\|\bar u - \bar u_h \circ g_h^{-1}\|_{L^2(\Gamma)}$. In §5 we prove that problems (P$_h$) realize a correct approximation of (P) in the sense that global solutions of (P$_h$) converge strongly to global solutions of (P) and the strict local solutions of (P) can be approximated by local solutions of problems (P$_h$). A crucial result in this section is the derivation of the estimates for the differences of states and of adjoint states defined in Ω and Ω_h, respectively. The reader is referred to Theorems 5.1 and 5.2 for the estimates in the spaces $H^s(\Omega_h)$, with $0 \le s \le 3/2$. One key point in this proof is the use of a modification of the Aubin-Nitsche argument to derive error estimates in the L^2 norm for finite element approximations. This approach used in the case of linear equations can be adapted to semilinear problems as it is shown. Finally, in §6 we derive the error estimates for the controls and the corresponding states and adjoint states.

In a forthcoming paper we analyze the case of a Dirichlet control problem. The reader is referred to [5] and [6] for the numerical approximation of this problem.

2. Assumptions and Preliminary Results. The following hypotheses are imposed on the data of problem (P).

(A1) Ω is an open, convex and bounded domain in $\mathbb{R}^2$, with the boundary Γ of class C^2. Moreover, we assume that $N > 0$ and $-\infty \le \alpha < \beta \le +\infty$.

(A2) $L : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ and $a : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ are Carathéodory functions of class C^2 with respect to the second variable, $L(\cdot, 0) \in L^1(\Omega)$, $a(\cdot, 0) \in L^\infty(\Omega)$ and for every $M > 0$ there exist a constant C_M such that for almost all $x \in \Omega$ and all $|y|, |y_i| \le M$, $i = 1, 2$, the following inequalities hold

$$(2.1)\qquad \left\{\begin{array}{l} \displaystyle\sum_{j=1}^{2}\left\{\left|\frac{\partial^j L}{\partial y^j}(x,y)\right| + \left|\frac{\partial^j a}{\partial y^j}(x,y)\right|\right\} \le C_M, \\ \displaystyle\left|\frac{\partial^2 L}{\partial y^2}(x,y_2) - \frac{\partial^2 L}{\partial y^2}(x,y_1)\right| + \left|\frac{\partial^2 a}{\partial y^2}(x,y_2) - \frac{\partial^2 a}{\partial y^2}(x,y_1)\right| \le C_M|y_2 - y_1|. \end{array}\right.$$

We also assume

$$(2.2)\qquad \left\{\begin{array}{l} \displaystyle\frac{\partial a}{\partial y}(x,y) \ge 0 \quad \text{for a.e.} \quad x \in \Omega \quad \text{and for all} \quad y \in \mathbb{R}, \\ \displaystyle\exists E \subset \Omega \text{ and } \Lambda > 0 \text{ such that } |E| > 0 \text{ and } \frac{\partial a}{\partial y}(x,y) \ge \Lambda \ \forall (x,y) \in E \times \mathbb{R}. \end{array}\right.$$

We observe that, by our assumptions *(A1)* and *(A2)*, for every $u \in L^2(\Gamma)$ the state equation (1.1) has a unique solution $y_u \in L^\infty(\Omega) \cap H^1(\Omega)$. The proof is standard and some estimates can be derived

$$(2.3)\qquad \|y_u\|_{H^1(\Omega)} + \|y\|_{L^\infty(\Omega)} \le C_E \left(\|a(\cdot,0)\|_{L^2(\Omega)} + \|u\|_{L^2(\Gamma)}\right).$$

Moreover, if $u \in H^{1/2}(\Gamma)$, then $y_u \in H^2(\Omega)$ and we have an analogous estimate with the $L^2(\Gamma)$-norm of u replaced by the $H^{1/2}(\Gamma)$-norm.

To assure the existence of a global optimal solution of problem (P) we need an additional hypothesis.

(A3) Either $\alpha, \beta \in \mathbb{R}$ or the following assumption holds

$$(2.4)\qquad L(x,y) \ge \psi_L(x) + \Lambda_L y^2, \text{ with } \psi_L \in L^1(\Omega) \text{ and } N + 4C_E^2 \min\{0, \Lambda_L\} > 0.$$

where C_E is as in (2.3). Indeed, if we take a minimizing sequence $\{u_k\}_{k=1}^\infty$ of problem (P), then either $\alpha, \beta \in \mathbb{R}$ and consequently $\{u_k\}_{k=1}^\infty$ is bounded in $L^\infty(\Gamma)$ or

$$J(u_k) \ge \int_\Omega \psi_L(x)\,dx + \Lambda_L \int_\Omega y_k^2(x)\,dx + \frac{N}{2}\|u_k\|^2_{L^2(\Gamma)}$$

$$\ge \int_\Omega \psi_L(x)\,dx + 2\min\{0,\Lambda_L\}C_E^2\left(\|a(\cdot,0)\|^2_{L^2(\Omega)} + \|u_k\|^2_{L^2(\Gamma)}\right) + \frac{N}{2}\|u_k\|^2_{L^2(\Gamma)}$$

$$= C + \left(\frac{N}{2} + 2\min\{0,\Lambda_L\}C_E^2\right)\|u_k\|^2_{L^2(\Gamma)},$$

which allows to conclude again that $\{u_k\}_{k=1}^\infty$ is bounded in $L^2(\Gamma)$. The remaining part of the proof is classical.

3. First and Second Order Optimality Conditions for (P). In this section we establish the first and second order optimality conditions for the local minimum of (P), which are necessary to derive error estimates when approximating (P) by (P_h). Since problem (P) is not necessarily convex, then it may have more than one global solution as well as some local solutions which are not global. The optimality system for a local solution is stated in the next theorem, where we also establish the regularity of the local minima.

THEOREM 3.1. *Let $\bar u$ be a local minimum of* (P). *Then $\bar u \in C^{0,1}(\Gamma)$ and there exist elements $\bar y, \bar\varphi \in W^{2,p}(\Omega)$, for every $1 \le p < +\infty$, such that*

$$\left\{\begin{array}{rcll} -\Delta \bar y + a(x,\bar y) & = & 0 & \text{in } \Omega, \\ \partial_\nu \bar y & = & \bar u & \text{on } \Gamma, \end{array}\right. \tag{3.1}$$

$$\left\{\begin{array}{rcll} -\Delta \bar\varphi + \dfrac{\partial a}{\partial y}(x,\bar y)\bar\varphi & = & \dfrac{\partial L}{\partial y}(x,\bar y) & \text{in } \Omega, \\ \partial_\nu \bar\varphi & = & 0 & \text{on } \Gamma, \end{array}\right. \tag{3.2}$$

$$\int_\Gamma (\bar\varphi(x) + N\bar u(x))(v(x) - \bar u(x))\, d\sigma(x) \ge 0 \quad \text{for all } \alpha \le v \le \beta. \tag{3.3}$$

Sketch of the proof. First, we note that $J : L^2(\Gamma) \longrightarrow \mathbb{R}$ is of class C^1 (in fact, it is of class C^2) and

$$J'(\bar u)v = \int_\Gamma (\bar\varphi(x) + N\bar u(x))v(x)\, d\sigma(x),$$

where $\bar\varphi \in L^\infty(\Omega) \cap H^1(\Omega)$ is the solution of (3.2) and $\bar y$ is the state associated to $\bar u$ and consequently the unique solution of (3.1) in $L^\infty(\Omega) \cap H^1(\Omega)$. The well known optimality condition

$$J'(\bar u)(v - \bar u) \ge 0 \quad \text{for all } \alpha \le v \le \beta,$$

along with the expression of J' lead to (3.3). Now (3.3) implies

$$\bar u(x) = \mathrm{Proj}_{[\alpha,\beta]}\left(-\frac{1}{N}\bar\varphi(x)\right) = \max\{\alpha, \min\{-\frac{1}{N}\bar\varphi(x), \beta\}\}. \tag{3.4}$$

From our assumption *(A2)* and the boundedness of $\bar y$ we have that

$$\frac{\partial L}{\partial y}(x,\bar y(x)), \frac{\partial a}{\partial y}(x,\bar y(x)) \in L^\infty(\Omega).$$

Therefore, we can use the elliptic regularity results (see Grisvard [7, Chapter 2]) to deduce that $\bar\varphi \in W^{2,p}(\Omega)$ for every $1 \le p < +\infty$. Moreover, since $W^{2,p}(\Omega) \subset C^1(\bar\Omega)$ for every $p > 2$, we get from (3.4) that $\bar u$ is Lipschitz in Γ. Finally, from (3.1) and using again the elliptic regularity results, we conclude that $\bar y \in W^{2,p}(\Omega)$ for every $1 \le p < +\infty$.

Let us observe that (3.3) is equivalent to $\bar\varphi + N\bar u = 0$ on Γ if $\alpha = -\infty$ and $\beta = +\infty$. In this case $\bar u = -\bar\varphi/N \in W^{2-1/p,p}(\Gamma)$ for all $1 \le p < +\infty$.

We finish this section by stating the second order optimality conditions. Given a local minimum $\bar u$, the associated cone of critical directions is defined by

$$C_{\bar u} = \{v \in L^2(\Gamma) \text{ satisfying (3.5) and such that } v(x) = 0 \text{ if } |\bar\varphi(x) + N\bar u(x)| > 0\},$$

$$v(x) = \begin{cases} \geq 0 & \text{if } \bar{u}(x) = \alpha, \\ \leq 0 & \text{if } \bar{u}(x) = \beta. \end{cases} \tag{3.5}$$

Then we have the following result.

THEOREM 3.2. *If $\bar{u}$ is a local minimum of problem* (P)*, then $J''(\bar{u})v^2 \geq 0$ for all $v \in C_{\bar{u}}$. Reciprocally, if $\bar{u}$ is a feasible control for problem* (P) *satisfying the first order optimality conditions* (3.1)-(3.3) *and the coercivity condition*

$$J''(\bar{u})v^2 > 0 \quad \forall v \in C_{\bar{u}} \setminus \{0\}, \tag{3.6}$$

then there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$J(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}\|_{L^2(\Gamma)} \leq J(u) \tag{3.7}$$

for every $\alpha \leq u \leq \beta$ such that $\|u - \bar{u}\|_{L^2(\Gamma)} < \varepsilon$.

For the details, the reader is referred to [2] and [4]. An important fact is that condition (3.6) holds if and only if

$$\exists \mu > 0 \text{ and } \vartheta > 0 \text{ such that } J''(\bar{u})v^2 \geq \mu\|v\|^2_{L^2(\Gamma)} \ \forall v \in C^{\vartheta}_{\bar{u}}, \tag{3.8}$$

where

$$C^{\vartheta}_{\bar{u}} = \{v \in L^2(\Gamma) \text{ satisfying (3.5) and } v(x) = 0 \text{ if } |\bar{\varphi}(x) + N\bar{u}(x)| > \vartheta\}.$$

4. Problem (P_h). In order to define the control problem (P_h) we consider a polygonal approximation of Ω. We fix a set of points $\{x_j\}_{j=1}^{N(h)} \subset \Gamma$, the nodes being ordered clockwise. We set

$$h_j = |x_{j+1} - x_j|, \quad h = \max_{1 \leq j \leq N(h)} h_j, \quad \tau_j = \frac{1}{h_j}(x_{j+1} - x_j),$$

where we denote $x_{N(h)+1} = x_1$. Γ_h is the polygonal line defined by the nodes $\{x_j\}_{j=1}^{N(h)}$ and Ω_h is the polygon delimited by Γ_h. Since Ω is convex, it is clear that $\Omega_h \subset \Omega$. For every $1 \leq j \leq N(h)$, $\widehat{x_j x_{j+1}}$ denotes the arc of Γ delimited by the points x_j and x_{j+1}. Then we have that $\Gamma = \cup_{j=1}^{N(h)} \widehat{x_j x_{j+1}}$ and $\Gamma_h = \cup_{j=1}^{N(h)} [x_j, x_{j+1}]$. For every $1 \leq j \leq N(h)$, ν_j represents the unit outward normal vector to Ω_h on the boundary edge (x_j, x_{j+1}).

Now we introduce a parametrization of Γ as follows

$$\psi_j : [0, h_j] \longrightarrow \widehat{x_j x_{j+1}} \subset \Gamma \text{ is defined by } \psi_j(t) = x_j + t\tau_j + \phi_j(t)\nu_j,$$

$\phi_j : [0, h_j] \longrightarrow [0, +\infty)$ is chosen such that $\psi_j(t) \in \Gamma$. It is evident that ϕ_j is uniquely defined. Since Ω is convex and Γ is of class C^2, the following properties hold

1. ϕ_j is of class C^2 and $\phi_j(0) = \phi_j(h_j) = 0$.
2. There exists a constant $C_\Gamma > 0$ such that $\phi_j(t) + h|\phi'_j(t)| \leq C_\Gamma h_j^2 \leq C_\Gamma h^2$ for all $t \in [0, h_j]$.

Finally, we define

$$g_h : \Gamma_h \longrightarrow \Gamma, \quad g_h|_{[x_j,x_{j+1}]}(x) = g_h|_{[x_j,x_{j+1}]}(x_j + t\tau_j) = x_j + t\tau_j + \phi_j(t)\nu_j = \psi_j(t).$$

Clearly g_h is one-to-one. We denote by $\nu(x)$ the unit outward normal vector to Γ at the point x and by $\tau(x)$ the unit tangent vector such that $\{\tau(x), \nu(x)\}$ is a direct reference system in $\mathbb{R}^2$. We can obtain the expressions for these vectors from the parametrization. If x is a point of the arc $\widehat{x_jx_{j+1}}$, then

$$\tau(x) = \frac{1}{\sqrt{1+\phi_j'(t)^2}}(\tau_j + \phi_j'(t)\nu_j) \quad \text{and} \quad \nu(x) = \frac{1}{\sqrt{1+\phi_j'(t)^2}}(\nu_j - \phi_j'(t)\tau_j).$$

From these expressions and the properties of ϕ_j we deduce that

$$|\nu(g_h(x)) - \nu_j| \le \sqrt{C_\Gamma^2 h^2 + 1} C_\Gamma h \quad \forall x \in [x_j, x_{j+1}],$$

the same inequality holds true for $|\tau(g_h(x)) - \tau_j|$. Since we are interested in the case of $h \to 0$, we can assume that $h < 1$ and then

$$\max\{|\tau(g_h(x)) - \tau_h(x)|, |\nu(g_h(x)) - \nu_h(x)|\} \le (C_\Gamma^2 + 1)h \quad \forall x \in \Gamma_h, \tag{4.1}$$

where $\tau_h(x) = \tau_j$ and $\nu_h(x) = \nu_j$ if $x \in (x_j, x_{j+1})$.

Given a function $v \in L^1(\Gamma)$, we have

$$\int_\Gamma v(x)\,d\sigma(x) = \sum_{j=1}^{N(h)} \int_0^{h_j} v(\psi_j(t))\sqrt{1+\phi_j'(t)^2}\,dt$$

and

$$\int_{\Gamma_h} v(g_h(x))\,d\sigma_h(x) = \sum_{j=1}^{N(h)} \int_0^{h_j} v(g_h(x_j + t\tau_j))\,dt = \sum_{j=1}^{N(h)} \int_0^{h_j} v(\psi_j(t))\,dt.$$

From these expressions we deduce that

$$\int_{\Gamma_h} |v(g_h(x))|\,d\sigma_h(x) \le \int_\Gamma |v(x)|\,d\sigma(x) \quad \forall v \in L^1(\Gamma) \tag{4.2}$$

and

$$\left|\int_\Gamma v(x)\,d\sigma(x) - \int_{\Gamma_h} v(g_h(x))\,d\sigma_h(x)\right| \le \sum_{j=1}^{N(h)} \int_0^{h_j} |v(\psi_j(t))||1 - \sqrt{1+\phi_j'(t)^2}|\,dt$$

$$\le C_\Gamma h^2 \sum_{j=1}^{N(h)} \int_0^{h_j} |v(\psi_j(t))|\,dt \le C_\Gamma h^2 \int_\Gamma |v(x)|\,d\sigma(x) \quad \forall v \in L^1(\Gamma). \tag{4.3}$$

We also have

$$\int_\Gamma v(x)\,d\sigma(x) = \int_{\Gamma_h} v(g_h(x))|Dg_h(x)\cdot\tau_h(x)|\,d\sigma_h(x) \quad \forall v \in L^1(\Gamma). \tag{4.4}$$

In the domain Ω_h defined above we consider the state equation (1.2) and the associated control problem (P_h) described in Introduction. Since we are interested in the behavior of the solutions of (P_h) when $h \to 0$, we can assume without any lost of generality that there exists $h_0 > 0$ such that the set $E \subset \Omega$, introduced in assumption *(A2)*, is also contained in Ω_h for every $h \le h_0$. Then assumptions *(A1)* and *(A2)* imply the existence of a unique solution $y_{h,u}$ of (1.2) in $H^1(\Omega_h)\cap L^\infty(\Omega_h)$ for every $u\in L^2(\Gamma_h)$. Moreover, the inequality (2.3) can be rewritten as follows

$$\|y_{h,u}\|_{H^1(\Omega_h)}+\|y\|_{L^\infty(\Omega_h)}\le C_E\left(\|a(\cdot,0)\|_{L^2(\Omega)}+\|u\|_{L^2(\Gamma_h)}\right)\quad\forall h\le h_0. \tag{4.5}$$

Since Ω_h is a convex polygonal domain we have that $y_{h,u}\in H^2(\Omega_h)$ whenever $u\in H^{1/2}(\Gamma_h)$; see, for instance, Grisvard [7, Chapter 4].

Arguing as in §2, we can prove that problem (P$_h$) has at least one global minimum for every $h\le h_0$. Furthermore, we have the optimality system analogous to (3.1)-(3.3).

Theorem 4.1. *Let $\bar u_h$ be a local minimum of* (P$_h$). *Then $\bar u_h\in H^1(\Gamma_h)$ and there exist elements $\bar y_h,\bar\varphi_h\in H^2(\Omega_h)$ such that*

$$\begin{cases} -\Delta\bar y_h+a(x,\bar y_h) &= 0 \quad \text{in } \Omega_h\\ \partial_\nu\bar y_h &= \bar u_h \quad \text{on } \Gamma_h\end{cases} \tag{4.6}$$

$$\begin{cases} -\Delta\bar\varphi_h+\dfrac{\partial a}{\partial y}(x,\bar y_h)\bar\varphi_h &= \dfrac{\partial L}{\partial y}(x,\bar y_h) \quad \text{in } \Omega_h\\ \partial_{\nu_h}\bar\varphi_h &= 0 \quad \text{on } \Gamma_h\end{cases} \tag{4.7}$$

$$\int_{\Gamma_h}(\bar\varphi_h(x)+N\bar u_h(x))(v_h(x)-\bar u_h(x))\,d\sigma_h(x)\ge 0\quad \text{for all}\ \ \alpha\le v_h\le\beta\ . \tag{4.8}$$

The proof of this theorem is the same as of Theorem 3.1 with the only one difference concerning the regularity of $(\bar u_h,\bar y_h,\bar\varphi_h)$. This difference is due to the lack of the regularity of Γ_h, which is not $C^{1,1}$ and thus the regularity results used in Theorem 3.1 are not valid. However, taking into account that Ω_h is convex, we can deduce that $\varphi_h\in H^2(\Omega_h)$; see Grisvard [7, Chapter 3]. Moreover, we have

$$\|\bar\varphi_h\|_{H^2(\Omega_h)}\le C\left(\|\frac{\partial a}{\partial y}(x,\bar y_h)\|_{L^2(\Omega_h)}+\|\frac{\partial L}{\partial y}(x,\bar y_h)\|_{L^2(\Omega_h)}\right),$$

where C is independent of h. Hence from (4.5) and assumption *(A2)* it follows that

$$\|\bar\varphi_h\|_{H^2(\Omega_h)}\le M_{\bar u_h}, \tag{4.9}$$

where $M_{\bar u_h}$ is a constant depending on $\|\bar u_h\|_{L^2(\Gamma_h)}$. Using (4.8) we get

$$\bar u_h(x)=\mathrm{Proj}_{[\alpha,\beta]}\left(-\frac{1}{N}\bar\varphi_h(x)\right)=\max\{\alpha,\min\{-\frac{1}{N}\bar\varphi_h(x),\beta\}\}, \tag{4.10}$$

which implies that $\bar u_h\in H^1(\Gamma_h)$, hence $\bar y_h\in H^2(\Omega_h)$ and

$$\|\bar y_h\|_{H^2(\Omega_h)}+\|\bar u_h\|_{H^1(\Gamma_h)}\le K_{\bar u_h}, \tag{4.11}$$

where once again $K_{\bar u_h}$ is a constant depending only on $\|\bar u_h\|_{L^2(\Gamma_h)}$ and independent of h.

If $-\infty<\alpha<\beta<+\infty$, then

$$\|\bar u_h\|_{L^2(\Gamma_h)}\le\max\{|\alpha|,|\beta|\}|\Gamma_h|^{1/2}\le\max\{|\alpha|,|\beta|\}|\Gamma|^{1/2}.$$

If $\alpha=-\infty$ or $\beta=+\infty$, then by (2.4) and the same argument as used at the end of §2 we get for all $u_h\in L^2(\Gamma_h)$ with $\alpha\le u_h\le\beta$

$$C+\left(\frac{N}{2}+2\min\{0,\Lambda_L\}C_E^2\right)\|\bar u_h\|^2_{L^2(\Gamma_h)}\le J_h(\bar u_h)\le J_h(u_h).$$

If $\bar{u}_h$ is a global solution of (P$_h$), then we can take $u_h \equiv c_{\alpha,\beta}$, with a constant $\alpha < c_{\alpha,\beta} < \beta$, and deduce from the above inequality, in view of (4.5), the boundedness of $\{\|\bar{u}_h\|_{L^2(\Gamma_h)}\}_{h\le h_0}$. In any case, by (4.9) and (4.11) there is a constant $K > 0$ such that

$$\|\bar{y}_h\|_{H^2(\Omega_h)} + \|\bar{\varphi}_h\|_{H^2(\Omega_h)} + \|\bar{u}_h\|_{H^1(\Gamma_h)} \le K \quad \forall h \le h_0. \tag{4.12}$$

When $\{\bar{u}_h\}_{h\le h_0}$ are just local minima of problems (P$_h$), the inequality (4.12) remains valid for $-\infty < \alpha < \beta < +\infty$ or for a bounded sequence $\{J_h(\bar{u}_h)\}_{h\le h_0}$, which is true provided $\{\|\bar{u}_h\|_{L^2(\Gamma_h)}\}_{h\le h_0}$ is bounded (cf. (4.5)).

5. Convergence Analysis. The goal of this section is to prove the convergence, in a sense to be defined later, of the solutions $\bar{u}_h$ of (P$_h$) to the solutions $\bar{u}$ of (P). We also analyze the approximation of local minima of (P) by local minima of problems (P$_h$). In order to carry out this analysis, first we compare the solutions of (1.1) and (1.2).

THEOREM 5.1. *Let $u \in H^{1/2}(\Gamma)$ and $u_h \in L^2(\Gamma_h)$, with*

$$\max\{\|u\|_{L^2(\Gamma)}, \|u_h\|_{L^2(\Gamma_h)}\} \le M. \tag{5.1}$$

Let $y_u \in H^2(\Omega)$ and $y_{h,u_h} \in H^{3/2}(\Omega_h)$ be the corresponding solutions of (1.1) *and* (1.2)*, respectively. Then there exists a constant $C_M > 0$ independent of h such that for all $0 \le s \le \frac{3}{2}$ the following estimate holds*

$$\|y_u - y_{h,u_h}\|_{H^s(\Omega_h)} \le C_M \left(\|u - u_h \circ g_h^{-1}\|_{L^2(\Gamma)} + h^{\frac{6-2s}{3}}[1 + \|u\|_{H^{1/2}(\Gamma)}]\right). \tag{5.2}$$

Proof. Let us introduce the intermediate problem

$$\left\{\begin{array}{rcll} -\Delta y_h + a(x, y_h) & = & 0 & \text{in } \Omega_h, \\ \partial_{\nu_h} y_h & = & u \circ g_h & \text{on } \Gamma_h. \end{array}\right. \tag{5.3}$$

Then we have

$$\|y_u - y_{h,u_h}\|_{H^s(\Omega_h)} \le \|y_u - y_h\|_{H^s(\Omega_h)} + \|y_h - y_{h,u_h}\|_{H^s(\Omega_h)}. \tag{5.4}$$

Let us estimate the second term of the right-hand side in (5.4). We set $\phi_h = y_h - y_{h,u_h}$. By substraction of the equations satisfied by y_h and y_{h,u_h} and using the mean value theorem, we get

$$\left\{\begin{array}{rcll} -\Delta \phi_h + \dfrac{\partial a}{\partial y}(x, w_h)\phi_h & = & 0 & \text{in } \Omega_h \\ \partial_{\nu_h} \phi_h & = & u \circ g_h - u_h & \text{on } \Gamma_h, \end{array}\right. \tag{5.5}$$

where $w_h = y_h + \theta_h(y_{h,u_h} - y_h)$ and $0 < \theta_h < 1$. From (5.5) and assumption (2.2) it follows that

$$\|\phi_h\|_{H^1(\Omega_h)} + \|\phi_h\|_{L^\infty(\Omega_h)} \le \|u \circ g_h^{-1} - u_h\|_{L^2(\Gamma_h)}.$$

In view of (5.1), we can apply (2.3) and (4.5) to obtain that

$$\|\frac{\partial a}{\partial y}(x, w_h)\|_{L^\infty(\Omega_h)} \le C_1$$

for some constant C_1 depending on M (cf. Assumption *(A2)*). Then we get

$$\|\phi_h\|_{H^{3/2}(\Omega_h)} \le C_2 \left(\|\Delta\phi_h\|_{L^2(\Omega_h)} + \|u \circ g_h - u_h\|_{L^2(\Gamma_h)}\right)$$

$$\le C_2 \left(C_1\|\phi_h\|_{L^2(\Omega_h)} + \|u \circ g_h - u_h\|_{L^2(\Gamma_h)}\right) \le C_3\|u \circ g_h - u_h\|_{L^2(\Gamma_h)},$$

see [9] for the first estimate. Now (4.2) combined with the above inequality lead to

$$\|y_h - y_{h,u_h}\|_{H^s(\Omega_h)} \le C_3\|u - u_h \circ g_h^{-1}\|_{L^2(\Gamma)} \text{ for all } 0 \le s \le \frac{3}{2}. \tag{5.6}$$

The remaining part of the proof is dedicated to derivation of the inequality

$$\|y_u - y_h\|_{H^s(\Omega_h)} \le Ch^{\frac{6-2s}{3}}[\|u\|_{H^{1/2}(\Gamma)} + 1] \text{ for all } 0 \le s \le \frac{3}{2}, \tag{5.7}$$

where C depends on the constant M given in (5.1). Thus (5.6) and (5.7) imply (5.2). The proof follows some steps. First, we consider the case $s = 3/2$, then by using the Aubin-Nitsche duality method we deduce the estimate for $s = 0$ and finally an appropriate interpolation inequality completes the proof.

Case 1: $s = 3/2$. Let us use again the letter ϕ_h to denote $\phi_h = y_u - y_h$. By substraction of the equations satisfied by y_u and y_h and by an application the mean value theorem we get

$$\left\{\begin{array}{rcll} -\Delta\phi_h + \dfrac{\partial a}{\partial y}(x, w_h)\phi_h &=& 0 & \text{in } \Omega_h \\ \partial_{\nu_h}\phi_h &=& \partial_{\nu_h}y_u - u \circ g_h & \text{on } \Gamma_h. \end{array}\right. \tag{5.8}$$

Using once again [9], we get

$$\|\phi_h\|_{H^{3/2}(\Omega_h)} \le C_3\|\partial_{\nu_h}y_u - u \circ g_h\|_{L^2(\Gamma_h)}$$

$$\le C_3 \left\{\|\nabla y_u \cdot \nu_h - (\nabla y_u \circ g_h) \cdot \nu_h\|_{L^2(\Gamma_h)}\right.$$

$$\left. + \|(\nabla y_u \circ g_h) \cdot \nu_h - (\nabla y_u \circ g_h) \cdot (\nu \circ g_h)\|_{L^2(\Gamma_h)}\right\}$$

$$\le C_3 \left\{\|\nabla y_u - \nabla y_u \circ g_h\|_{L^2(\Gamma_h)} + \|\nabla y_u \circ g_h\|_{L^2(\Gamma_h)}\|\nu_h - \nu \circ g_h\|_{L^2(\Gamma_h)}\right\}.$$

From [1, Lemma 1] we have

$$\|w - w \circ g_h\|_{L^2(\Gamma_h)} \le Ch^r\|w\|_{H^r(\Omega)} \text{ for all } 1 \le r \le 2. \tag{5.9}$$

Using this inequality with $r = 1$ and $w = \nabla y$ in the above estimate for ϕ_h along with (4.1) we get

$$\|y_u - y_h\|_{H^{3/2}(\Omega_h)} \le C_4h\|y_u\|_{H^2(\Omega)} \le C_5h[\|u\|_{H^{1/2}(\Gamma)} + 1], \tag{5.10}$$

where C_5 depends on the $L^2(\Omega)$-norm of $\frac{\partial a}{\partial y}(x, y_u)y_u$. By using (2.3) and assumption *(A2)* we get that the norm can be estimated by a constant depending on M, which implies that C_5 depends on M as well.

case 2: $s=0$. Let us define the function $\mu_h \in L^\infty(\Omega_h)$ by

$$\mu_h(x) = \begin{cases} \dfrac{a(x,y_u(x)) - a(x,y_h(x))}{y_u(x)-y_h(x)} & \text{if } y_u(x) \neq y_h(x) \\ 0 & \text{otherwise.} \end{cases}$$

Let $f \in L^2(\Omega_h)$ be arbitrary. We extend f and μ_h to Ω by zero and we define $z \in H^2(\Omega)$ and $z_h \in H^2(\Omega_h)$ as the solutions of the problems

$$\left\{ \begin{array}{rcll} -\Delta z + \mu_h(x) z & = & f & \text{in } \Omega, \\ \partial_\nu z & = & 0 & \text{on } \Gamma, \end{array} \right. \tag{5.11}$$

and

$$\left\{ \begin{array}{rcll} -\Delta z_h + \mu_h(x) z_h & = & f & \text{in } \Omega_h, \\ \partial_{\nu_h} z_h & = & 0 & \text{on } \Gamma_h. \end{array} \right. \tag{5.12}$$

Taking the difference of the equations (5.11) and (5.12) and arguing as above we get

$$\|z - z_h\|_{H^{3/2}(\Omega_h)} \leq C_6 \|\partial_{\nu_h} z\|_{L^2(\Gamma_h)} = C_6 \|\nabla z \cdot \nu_h - (\nabla z \circ g_h)\cdot(\nu \circ g_h)\|_{L^2(\Gamma_h)}$$

$$\leq C_6 \left\{ \|\nabla z - \nabla z \circ g_h\|_{L^2(\Gamma_h)} + \|\nabla z \circ g_h\|_{L^2(\Gamma_h)} \|\nu_h - \nu \circ g_h\|_{L^2(\Gamma_h)} \right\}$$

$$\leq C_7 h \|z\|_{H^2(\Omega)} \leq C_8 h \|f\|_{L^2(\Omega_h)}. \tag{5.13}$$

Now multiplying equation (5.12) by $y_u - y_h$, integrating by parts and using the equations satisfied by y_u and y_h we get

$$\int_{\Omega_h} f(y_u - y_h)\,dx = \int_{\Omega_h} \{\nabla z_h(\nabla y_u - \nabla y_h) + [a(x,y_u) - a(x,y_h)] z_h\}\,dx$$

$$= \int_{\Omega_h} \{(\nabla z_h - \nabla z)(\nabla y_u - \nabla y_h) + [a(x,y_u) - a(x,y_h)](z_h - z)\}\,dx$$

$$+ \int_{\Omega_h} \{\nabla z \nabla y_u + a(x,y_u) z\}\,dx - \int_{\Omega_h} \{\nabla z \nabla y_h + a(x,y_h) z\}\,dx$$

$$\leq \|z_h - z\|_{H^1(\Omega_h)} \|y_u - y_h\|_{H^1(\Omega_h)} - \int_{\Omega \setminus \Omega_h} \{\nabla z \nabla y_u + a(x,y_u) z\}\,dx$$

$$+ \int_\Gamma u z\,d\sigma - \int_{\Gamma_h} (u \circ g_h) z\,d\sigma_h. \tag{5.14}$$

From (5.10) and (5.13) we obtain

$$\|z_h - z\|_{H^1(\Omega_h)} \|y_u - y_h\|_{H^1(\Omega_h)} \leq C_5 C_8 h^2 [\|u\|_{H^{1/2}(\Gamma)} + 1] \|f\|_{L^2(\Omega_h)}. \tag{5.15}$$

To estimate the second term on the right-hand side of (5.14) we use the inequality (see [1, Lemma 2])

$$\|w\|_{L^2(\Omega\setminus\Omega_h)} \leq Ch\|w\|_{H^1(\Omega)}. \tag{5.16}$$

On the other hand, recalling that $0 \leq \phi_j(t) \leq C_\Gamma h^2$ for every $1 \leq j \leq N(h)$, we get the well known estimate

$$|\Omega \setminus \Omega_h| \leq Ch^2. \tag{5.17}$$

From (5.16) and (5.17) we get

$$\begin{aligned}
&\left|\int_{\Omega\setminus\Omega_h} \{\nabla z \nabla y_u + a(x,y_u)z\}\, dx\right| \\
&\leq \|\nabla z\|_{L^2(\Omega\setminus\Omega_h)}\|\nabla y_u\|_{L^2(\Omega\setminus\Omega_h)} + \|a(x,y_u)\|_{L^2(\Omega\setminus\Omega_h)}\|z\|_{L^2(\Omega\setminus\Omega_h)} \\
&\leq Ch^2\|z\|_{H^2(\Omega)}\|y_u\|_{H^2(\Omega)} + \sqrt{|\Omega\setminus\Omega_h|}\|a(x,y_u)\|_{L^\infty(\Omega)}Ch\|z\|_{H^1(\Omega)} \\
&\leq C_9 h^2[\|y_u\|_{H^2(\Omega)} + 1]\|f\|_{L^2(\Omega_h)} \leq C_{10}h^2[\|u\|_{H^{1/2}(\Gamma)} + 1]\|f\|_{L^2(\Omega_h)},
\end{aligned} \tag{5.18}$$

where C_{10} depends on the constant M given by (5.1)

Finally, we estimate the last term of (5.14) by using (4.2), (4.3), (5.1) and (5.9)

$$\begin{aligned}
&\left|\int_\Gamma uz\, d\sigma - \int_{\Gamma_h} (u\circ g_h)z\, d\sigma_h\right| \leq \int_{\Gamma_h} |(u\circ g_h)(z\circ g_h - z)|\, d\sigma_h + C_\Gamma h^2 \int_\Gamma |uz|\, d\sigma \\
&\leq \|u\circ g_h\|_{L^2(\Gamma_h)}\|z\circ g_h - z\|_{L^2(\Gamma_h)} + C_\Gamma h^2\|u\|_{L^2(\Gamma)}\|z\|_{L^2(\Gamma)} \\
&\leq C_{11}h^2\|u\|_{L^2(\Gamma)}\|z\|_{H^2(\Omega)} \leq C_{12}Mh^2\|f\|_{L^2(\Omega_h)}.
\end{aligned} \tag{5.19}$$

Now, from (5.14), (5.15), (5.18) and (5.19) we deduce

$$\|y_u - y_h\|_{L^2(\Omega_h)} \leq Ch^2[\|u\|_{H^{1/2}(\Gamma)} + 1], \tag{5.20}$$

where C depends on M, but it is independent of h.

Case 3: $0 < s < 3/2$. This case can be obtained from Case 1 combined with Case 2 and the following interpolation inequality

$$\|w\|_{H^s(\Omega_h)} \leq \varepsilon\|w\|_{H^{3/2}(\Omega_h)} + K\varepsilon^{-\frac{2s}{3-2s}}\|w\|_{L^2(\Omega_h)} \tag{5.21}$$

which holds for any $\varepsilon > 0$; see [7, Theorem 1.4.3.3]. By setting $\varepsilon = h^{(3-2s)/3}$ in (5.21) and using (5.10) and (5.20), we deduce (5.7). □

The next step in our analysis is comparison of the adjoint state equations corresponding to y_u and y_{h,u_h}. More precisely, we introduce the adjoint states $\varphi_u \in H^2(\Omega)$ and $\varphi_{h,u_h} \in H^2(\Omega_h)$ as the solutions of the equations

$$\left\{\begin{array}{rcll} -\Delta\varphi_u + \dfrac{\partial a}{\partial y}(x,y_u)\varphi_u &=& \dfrac{\partial L}{\partial y}(x,y_u) & \text{in } \Omega, \\ \partial_\nu\varphi_u &=& 0 & \text{on } \Gamma, \end{array}\right. \tag{5.22}$$

and

$$\left\{\begin{array}{rcll} -\Delta\varphi_{h,u_h} + \dfrac{\partial a}{\partial y}(x, y_{h,u_h})\varphi_{h,u_h} & = & \dfrac{\partial L}{\partial y}(x, y_{h,u_h}) & \text{in } \Omega_h, \\ \partial_{\nu_h}\varphi_{h,u_h} & = & 0 & \text{on } \Gamma_h. \end{array}\right. \tag{5.23}$$

Then we have the following estimates.

THEOREM 5.2. *Let (u, y_u) and (u_h, y_{h,u_h}) be as in Theorem 5.1. Let $\varphi_u \in H^2(\Omega)$ and $\varphi_{h,u_h} \in H^2(\Omega_h)$ be the corresponding solutions of* (5.22) *and* (5.23)*, respectively. Then there exists a constant $C_M > 0$ independent of h such that for all $0 \le s \le \frac{3}{2}$ the following estimate holds*

$$\|\varphi_u - \varphi_{h,u_h}\|_{H^s(\Omega_h)} \le C_M \left(\|u - u_h \circ g_h^{-1}\|_{L^2(\Gamma)} + h^{\frac{6-2s}{3}}[1 + \|u\|_{H^{1/2}(\Gamma)}]\right). \tag{5.24}$$

Proof. We follow the steps of the proof of Theorem 5.1, with some simplifications because now the equations are linear and the boundary conditions are homogeneous. To estimate $\varphi_u - \varphi_{h,u_h}$ we use estimates (5.2). Let us consider $\varphi_h \in H^2(\Omega_h)$ given by a solution of

$$\left\{\begin{array}{rcll} -\Delta\varphi_h + \dfrac{\partial a}{\partial y}(x, y_u)\varphi_h & = & \dfrac{\partial L}{\partial y}(x, y_u) & \text{in } \Omega_h \\ \partial_{\nu_h}\varphi_h & = & 0 & \text{on } \Gamma_h \end{array}\right. \tag{5.25}$$

From assumption *(A2)* and estimates (5.2) we deduce the existence of a constant $C_1 > 0$ depending on M such that

$$\|\frac{\partial L}{\partial y}(x, y_u) - \frac{\partial L}{\partial y}(x, y_{h,u_h})\|_{L^2(\Omega_h)} + \|[\frac{\partial a}{\partial y}(x, y_u) - \frac{\partial a}{\partial y}(x, y_{h,u_h})]\varphi_{h,u_h}\|_{L^2(\Omega_h)}$$

$$\le C_1\|y_u - y_{h,u_h}\|_{L^2(\Omega_h)} \le C_2\left(\|u - u_h \circ g_h^{-1}\|_{L^2(\Gamma)} + h^2[1 + \|u\|_{H^{1/2}(\Gamma)}]\right). \tag{5.26}$$

From (5.23), (5.25) and (5.26) we obtain

$$\|\varphi_h - \varphi_{h,u_h}\|_{H^{3/2}(\Omega_h)} \le C_3\|\Delta(\varphi_h - \varphi_{h,u_h})\|_{L^2(\Omega_h)}$$

$$\le C_4\left(\|u - u_h \circ g_h^{-1}\|_{L^2(\Gamma)} + h^2[1 + \|u\|_{H^{1/2}(\Gamma)}]\right). \tag{5.27}$$

The remaining part of the proof is devoted to the derivation of the following estimate

$$\|\varphi_u - \varphi_h\|_{H^s(\Omega_h)} \le Ch^{\frac{6-2s}{3}}, \tag{5.28}$$

since (5.27) and (5.28) imply (5.24).

We start with the case of $s = 3/2$. To this end, we define $\phi_h = \varphi_u - \varphi_h$. From (5.22) and (5.25) we get

$$\left\{\begin{array}{rcll} -\Delta\phi_h + \dfrac{\partial a}{\partial y}(x, y_u)\phi_h & = & 0 & \text{in } \Omega_h, \\ \partial_{\nu_h}\phi_h & = & \partial_{\nu_h}\varphi_u & \text{on } \Gamma_h. \end{array}\right. \tag{5.29}$$

Then we have

$$\|\varphi_u - \varphi_h\|_{H^{3/2}(\Omega_h)} = \|\phi_h\|_{H^{3/2}(\Omega_h)} \le C_1 \|\partial_{\nu_h}\varphi_u\|_{L^2(\Gamma_h)} \le C_2 h, \tag{5.30}$$

where the estimate for $\partial_{\nu_h}\varphi_u$ is obtained in the same way as for $\partial_{\nu_h} z$ in (5.13).

Now, we prove (5.28) for $s = 0$. To apply the Aubin-Nitsche duality method we define for every $f \in L^2(\Omega)$ vanishing in $\Omega \setminus \Omega_h$ the functions $z \in H^2(\Omega)$ and $z_h \in H^2(\Omega_h)$ given by solutions of the problems

$$\left\{ \begin{array}{rcll} -\Delta z + \dfrac{\partial a}{\partial y}(x, y_u) z & = & f & \text{in } \Omega, \\ \partial_\nu z & = & 0 & \text{on } \Gamma. \end{array} \right. \tag{5.31}$$

and

$$\left\{ \begin{array}{rcll} -\Delta z_h + \dfrac{\partial a}{\partial y}(x, y_u) z_h & = & f & \text{in } \Omega_h, \\ \partial_{\nu_h} z_h & = & 0 & \text{on } \Gamma_h. \end{array} \right. \tag{5.32}$$

As in (5.13) we get

$$\|z - z_h\|_{H^{3/2}(\Omega_h)} \le Ch\|f\|_{L^2(\Omega_h)}. \tag{5.33}$$

The same arguments as in the proof of Theorem 5.1, in view of (5.30) and (5.33), lead to

$$\|\varphi_u - \varphi_h\|_{L^2(\Omega_h)} \le Ch^2, \tag{5.34}$$

where C depends on M.

Finally, (5.28) is proved for $0 < s < 3/2$ in the same way as in Theorem 5.1, using the inequality (5.21) with $\varepsilon = h^{(3-2s)/3}$ along with inequalities (5.30) and (5.34). ◻

Corollary 5.3. *Under the assumptions of Theorem 5.2 the following inequality holds*

$$\|\varphi_u - \varphi_{h,u_h}\|_{L^2(\Gamma_h)} \le C_M \left(\|u - u_h \circ g_h^{-1}\|_{L^2(\Gamma)} + h^{5/3}[1 + \|u\|_{H^{1/2}(\Gamma)}] \right) \tag{5.35}$$

for a constant C_M depending on M but independent of h.

Proof. Using the function φ_h defined in (5.25) and inequality (5.27) we get

$$\|\varphi_u - \varphi_{h,u_h}\|_{L^2(\Gamma_h)} \le \|\varphi_u - \varphi_h\|_{L^2(\Gamma_h)} + \|\varphi_h - \varphi_{h,u_h}\|_{L^2(\Gamma_h)}$$

$$\le \|\varphi_u - \varphi_h\|_{L^2(\Gamma_h)} + C\left(\|u - u_h \circ g_h^{-1}\|_{L^2(\Gamma)} + h^2[1 + \|u\|_{H^{1/2}(\Gamma)}] \right). \tag{5.36}$$

According to [7, Theorem 1.5.1.10] we have

$$\|\varphi_u - \varphi_h\|^2_{L^2(\Gamma_h)} \le K\left\{ \varepsilon^{1/2}\|\nabla(\varphi_u - \varphi_h)\|^2_{L^2(\Omega_h)} + \varepsilon^{-1/2}\|\varphi_u - \varphi_h\|^2_{L^2(\Omega_h)} \right\}. \tag{5.37}$$

Taking $s = 1$ in (5.28) and $\varepsilon = h^{4/3}$ in (5.37) it follows

$$\|\varphi_u - \varphi_h\|_{L^2(\Gamma_h)} \le Ch^{5/3}. \tag{5.38}$$

Finally (5.36) and (5.38) lead to (5.35) ◻

The remaining part of the section is devoted to the study of the convergence of solutions of (P_h) to the solutions of (P) with $h \to 0$. First, we prove that the solutions of (P_h) converge to solutions of (P). Since (P) is not convex, we are also interested in an inverse property: if a given local minimum of (P) can be approximated by local minima of problems (P_h). This question is positively answered in this section. Let us start with a theorem which provides a precise meaning for the convergence of controls. We recall that problems (P_h) admit at least one solution for each h small enough (see the comments before Theorem 4.1).

THEOREM 5.4. *Let $\bar{u}_h$ be a solution of problem (P_h) for $h \le h_0$. Then $\{\bar{u}_h \circ g_h^{-1}\}_{0<h\le h_0}$ is a bounded family in $H^1(\Gamma)$. If $\bar{u}$ is a weak limit for a subsequence, denoted in the same way, i.e. $\bar{u}_h \circ g_h^{-1} \to \bar{u}$ weakly in $H^1(\Gamma)$ with $h \to 0$, then $\bar{u}$ is a solution of problem (P). Moreover*

$$\lim_{h\to 0} \|\bar{y} - \bar{y}_h\|_{H^{3/2}(\Omega_h)} = 0 \quad \textit{and} \quad \lim_{h\to 0} J_h(\bar{u}_h) \to J(\bar{u}),$$

where $\bar{y}$ and $\bar{y}_h$ denote the solutions of (1.1) and (1.2) corresponding to $\bar{u}$ and $\bar{u}_h$, respectively.

Proof. The boundedness of $\{\bar{u}_h \circ g_h^{-1}\}_{0<h\le h_0}$ is an immediate consequence of (4.12). Let us prove the convergence of $\{J_h(\bar{u}_h)\}$. We denote by $\bar{y}_h$ and $\bar{y}$ the states associated to $\bar{u}_h$ and $\bar{u}$ respectively. Once again (4.12) implies that $\|\bar{u}_h\|_{L^2(\Gamma_h)} \le K$ for every $h \le h_0$. Then we can use the estimates (5.2) with $s = 3/2$ to get for $h \to 0$

$$\|\bar{y} - \bar{y}_h\|_{H^{3/2}(\Omega_h)} \le C_M \left(\|\bar{u} - \bar{u}_h \circ g_h^{-1}\|_{L^2(\Gamma)} + h[1 + \|u\|_{H^{1/2}(\Gamma)}]\right) \to 0.$$

This convergence imply also that $\|\bar{y} - \bar{y}_h\|_{C(\bar{\Omega})} \to 0$. Then we have by assumption *(A2)*

$$\left|\int_{\Omega_h} L(x, \bar{y}(x))\, dx - \int_{\Omega_h} L(x, \bar{y}_h(x))\, dx\right| \le C_1 \int_{\Omega_h} |\bar{y}(x) - \bar{y}_h(x)|\, dx \to 0.$$

On the other hand, it is obvious that

$$\lim_{h\to 0} \int_{\Omega\setminus\Omega_h} L(x, \bar{y}(x))\, dx = 0.$$

Finally, from (4.3) and by the strong convergence $\bar{u}_h \circ g_h^{-1} \to \bar{u}$ in $L^2(\Gamma)$ we obtain

$$\left|\int_\Gamma \bar{u}^2(x)\, d\sigma(x) - \int_{\Gamma_h} \bar{u}_h^2(x)\, d\sigma_h(x)\right|$$

$$\le \left|\int_\Gamma \bar{u}^2(x)\, d\sigma(x) - \int_\Gamma \bar{u}_h^2(g_h^{-1}(x))\, d\sigma(x)\right| + Ch^2 \to 0.$$

Collecting all these estimates we deduce the convergence $J_h(\bar{u}_h) \to J(\bar{u})$.

Let us show that $\bar{u}$ is a solution of (P). First we select an element $u \in H^{1/2}(\Gamma)$ such that $\alpha \le u \le \beta$ and we prove that $J(\bar{u}) \le J(u)$. Indeed, it is clear that $u \circ g_h$ is a feasible control for (P_h), consequently $J_h(\bar{u}_h) \le J_h(u \circ g_h)$. Furthermore, if we denote by y_h the state associated to $u \circ g_h$, then (5.2) implies that

$$\|y_u - y_h\|_{H^{3/2}(\Omega_h)} \le Ch[1 + \|u\|_{H^{1/2}(\Gamma)}].$$

This along with (4.3) imply the convergence $J_h(u \circ g_h) \to J(u)$. Thus, we have

$$J(\bar{u}) = \lim_{h\to 0} J_h(\bar{u}_h) \le \lim_{h\to 0} J_h(u \circ g_h) = J(u).$$

Finally, let us take $u \in L^2(\Gamma)$, with $\alpha \le u \le \beta$. There exists a sequence $\{u_k\}_{k=1}^{\infty} \subset H^{1/2}(\Gamma)$ such that $u_k \to u$ in $L^2(\Gamma)$. Setting $\hat{u}_k = \mathrm{Proj}_{[\alpha,\beta]}(u_k)$, we have that $\{\hat{u}_k\}_{k=1}^{\infty}$ is still strongly convergent to u in $L^2(\Gamma)$ and $\hat{u}_k \in H^{1/2}(\Gamma)$ is a feasible control for (P) for every k, then $J(\bar{u}) \le J(\hat{u}_k)$ for every k. Now passing to the limit we obtain $J(\bar{u}) \le J(u)$. Since u is an arbitrary feasible control of (P), this completes the proof. $\square$

Now we consider the approximation of local minima of (P) by local minima of problems (P_h). First let us say that whenever we speak about a local minimum it must be understood as a local minimum in the sense of L^2, more precisely it is the minimum among all feasible controls in a L^2-ball centered at the specific solution.

THEOREM 5.5. *Let $\bar{u}$ be a strict local minimum of* (P)*, then there exists a family $\{\bar{u}_h\}$ such that each $\bar{u}_h$ is a local minimum of* (P_h) *and $\bar{u}_h \circ g_h^{-1} \rightharpoonup \bar{u}$ weakly in $H^1(\Gamma)$.*

Proof. Let $\varepsilon > 0$ be such that $\bar{u}$ is the only global solution of problem

$$(\mathrm{P}_\varepsilon)\begin{cases} \min J(u) = \displaystyle\int_\Omega L(x, y_u(x))\,dx + \frac{N}{2}\int_\Gamma u^2(x)\,d\sigma(x) \\ \text{subject to } \ (y_u, u) \in (L^\infty(\Omega) \cap H^1(\Omega)) \times L^2(\Gamma), \\ \alpha \le u(x) \le \beta \quad \text{for a.e. } \ x \in \Gamma \ \text{ and } \ \|u - \bar{u}\|_{L^2(\Gamma)} \le \varepsilon. \end{cases}$$

Now for every $h \le h_0$ we consider the problems

$$(\mathrm{P}_{h\varepsilon})\begin{cases} \min J_h(u) = \displaystyle\int_{\Omega_h} L(x, y_{h,u}(x))\,dx + \frac{N}{2}\int_{\Gamma_h} u^2(x)\,d\sigma_h(x) \\ \text{subject to } \ (y_{h,u}, u) \in (L^\infty(\Omega_h) \cap H^1(\Omega_h)) \times L^2(\Gamma_h), \\ \alpha \le u(x) \le \beta \quad \text{for a.e. } \ x \in \Gamma_h \ \text{ and } \ \|u \circ g_h^{-1} - \bar{u}\|_{L^2(\Gamma)} \le \varepsilon. \end{cases}$$

It is obvious that $\bar{u} \circ g_h$ is a feasible control for every problem ($\mathrm{P}_{h\varepsilon}$), therefore, there exists at least one solution $u_{h\varepsilon}$ of ($\mathrm{P}_{h\varepsilon}$). Let us prove that $u_{h\varepsilon} \circ g_h^{-1} \to \bar{u}$ weakly in $H^1(\Gamma)$ with $h \to 0$.

Since $\{u_{h\varepsilon} \circ g_h^{-1}\}_{0<h\le h_0}$ is bounded in $L^2(\Gamma)$, we can take a subsequence, denoted in the same manner, and an element $\tilde{u} \in L^2(\Gamma)$ such that $u_{h\varepsilon} \circ g_h^{-1} \rightharpoonup \tilde{u}$ weakly in $L^2(\Gamma)$ with $h \to 0$. Let us denote by $y_{h\varepsilon} \in H^{3/2}(\Omega_h)$ the state associated to $u_{h\varepsilon}$ and consider an extension of $y_{h\varepsilon}$ to Ω, still denoted by $y_{h\varepsilon}$, such that

$$\|y_{h\varepsilon}\|_{H^{3/2}(\Omega)} \le C\|y_{h\varepsilon}\|_{H^{3/2}(\Omega_h)} \quad \forall h.$$

The boundedness of $\{u_{h\varepsilon} \circ g_h^{-1}\}_{0<h\le h_0}$ in $L^2(\Gamma)$ implies that $\{y_{h\varepsilon}\}$ is bounded in $H^{3/2}(\Omega)$. Therefore, by taking a subsequence, we can assume that

$$y_{h\varepsilon} \rightharpoonup \tilde{y} \ \text{ in } H^{3/2}(\Omega) \ \text{ and } \ u_{h\varepsilon} \circ g_h^{-1} \rightharpoonup \tilde{u} \ \text{ in } L^2(\Gamma).$$

Using the compactness of the imbeddings $H^{3/2}(\Omega) \subset H^1(\Omega)$ and $H^{3/2}(\Omega) \subset L^\infty(\Omega)$, it is easy to prove that $\tilde{y}$ is the solution of (1.1) associated to the control $\tilde{u}$. On the other hand, each $u_{h\varepsilon} \circ g_h^{-1}$ is a feasible control for (P_ε) and the set of feasible controls for this problem is convex and closed in $L^2(\Gamma)$, consequently $\tilde u$ is also a feasible control for (P_ε). From (5.2), the strong convergence $y_{h\varepsilon} \to \tilde y$ in $L^\infty(\Omega)$, the weak convergence $u_{h\varepsilon}\circ g_h^{-1} \rightharpoonup \tilde u$, (4.3) and the fact that $u_{h\varepsilon}$ is a solution of $(\mathrm{P}_{h\varepsilon})$ and $\bar u\circ g_h^{-1}$ is feasible for (P_ε) we get

$$J(\tilde u) \le \liminf_{h\to 0} J_h(u_{h\varepsilon}) \le \liminf_{h\to 0} J_h(\bar u\circ g_h^{-1}) \le \limsup_{h\to 0} J_h(\bar u\circ g_h^{-1}) = J(\bar u).$$

The fact that $\bar u$ is the unique solution of (P_ε) and the above inequalities lead to $\tilde u = \bar u$ and $J_h(u_{h\varepsilon}) \to J(\bar u)$, which implies

$$\lim_{h\to 0}\int_{\Gamma_h} u_{h\varepsilon}^2(x)\,d\sigma_h(x) = \int_\Gamma \bar u^2(x)\,d\sigma(x).$$

Using once again (4.3) we have

$$\lim_{h\to 0}\int_{\Gamma} (u_{h\varepsilon}\circ g_h^{-1})^2(x)\,d\sigma(x) = \int_\Gamma \bar u^2(x)\,d\sigma(x).$$

This identity and the weak convergence imply the strong convergence $u_{h\varepsilon}\circ g_h^{-1} \to \bar u$ in $L^2(\Gamma)$. A first consequence of this strong convergence is that the constraint $\|u\circ g_h^{-1} - \bar u\|_{L^2(\Gamma)} \le \varepsilon$ is not active for the elements $u_{h\varepsilon}$ if h small enough. Therefore, $u_{h\varepsilon}$ is a local minimum of problem (P_h) for h small enough. Since $\{\|u_{h\varepsilon}\|_{L^2(\Gamma_h)}\}$ is bounded, then we can use (4.12) and conclude that $\{u_{h\varepsilon}\circ g_h^{-1}\}$ is bounded in $H^1(\Gamma)$ and hence $u_{h\varepsilon}\circ g_h^{-1} \rightharpoonup \bar u$ weakly in $H^1(\Gamma)$ with $h\to 0$. ∎

6. Error Estimates. In this section we assume that $\bar u_h$ is a local minimum of (P_h), for every $h \le h_0$, such that $\bar u_h \circ g_h^{-1}$ converges weakly in $H^1(\Gamma)$ to a local minimum $\bar u$ of (P) with $h \to 0$; see Theorems 5.4 and 5.5. The goal of this section is to derive estimates of $\|\bar u - \bar u_h\circ g_h^{-1}\|_{L^2(\Gamma)}$, which are established in the following theorem.

THEOREM 6.1. *Let $\bar u$ and $\bar u_h$ be as above and let us denote by $\bar y, \bar y_h$ and $\bar\varphi, \bar\varphi_h$ the states and adjoint states associated to $\bar u$ and $\bar u_h$, respectively. Let us assume that the second order sufficient optimality condition* (3.6) *is fulfilled. Then there exists a constant C independent of h such that the following estimates hold*

(6.1) $\|\bar u - \bar u_h\circ g_h^{-1}\|_{L^2(\Gamma)} \le Ch^{5/3}$,

(6.2) $\|\bar y - \bar y_h\|_{H^s(\Omega_h)} + \|\bar\varphi - \bar\varphi_h\|_{H^s(\Omega_h)} \le Ch^{\min\{5,6-2s\}/3}$ *for all* $0 \le s \le \dfrac{3}{2}$.

Proof. By taking $v = \bar u_h\circ g_h^{-1}$ in (3.3) and $v_h = \bar u\circ g_h$ in (4.8) we get

$$J'(\bar u)(\bar u_h\circ g_h^{-1} - \bar u) = \int_\Gamma (\bar\varphi + N\bar u)(\bar u_h\circ g_h^{-1} - \bar u)\,d\sigma \ge 0 \tag{6.3}$$

and

$$J_h'(\bar u_h)(\bar u\circ g_h - \bar u_h) = \int_{\Gamma_h} (\bar\varphi_h + N\bar u_h)(\bar u\circ g_h - \bar u_h)\,d\sigma_h \ge 0. \tag{6.4}$$

We rewrite inequality (6.4) as follows

$$J'(\bar u_h\circ g_h^{-1})(\bar u - \bar u_h\circ g_h^{-1}) + [J_h'(\bar u_h)(\bar u\circ g_h - \bar u_h) - J'(\bar u_h\circ g_h^{-1})(\bar u - \bar u_h\circ g_h^{-1})] \ge 0. \tag{6.5}$$

From (6.3) and (6.5) we obtain

$$[J'(\bar{u}_h \circ g_n^{-1}) - J'(\bar{u})](\bar{u}_h \circ g_h^{-1} - \bar{u}) \le J'_h(\bar{u}_h)(\bar{u} \circ g_h - \bar{u}_h) - J'(\bar{u}_h \circ g_h^{-1})(\bar{u} - \bar{u}_h \circ g_h^{-1}).$$

By applying the mean value theorem we obtain the existence of an element $v_h = \bar{u} + \theta_h(\bar{u}_h \circ g_n^{-1} - \bar{u})$ such that

$$J''(v_h)(\bar{u}_h \circ g_h^{-1} - \bar{u})^2 \le J'_h(\bar{u}_h)(\bar{u} \circ g_h - \bar{u}_h) - J'(\bar{u}_h \circ g_h^{-1})(\bar{u} - \bar{u}_h \circ g_h^{-1}). \tag{6.6}$$

This inequality plays the central role in the derivation of (6.1). The proof is divided in two parts. First we use the second order optimality condition (3.6), or more precisely its equivalent formulation (3.8) to estimate the left hand side of (6.6) from below. In the second part we estimate the right-hand side in terms of h from above. The inequality (6.2) is an immediate consequence of (6.1) combined with the estimates (5.2) and (5.24).

Lower Bounds for (6.6). Let us prove that $\bar{u}_h \circ g_h^{-1} - \bar{u} \in C_{\bar{u}}^{\vartheta}$ for every h small enough. Indeed, $\bar{u}_h \circ g_h^{-1} - \bar{u}$ satisfies obviously conditions (3.5). Let us check that $(\bar{u}_h \circ g_h^{-1} - \bar{u})(x) = 0$ at the points x where $|\bar{\varphi}(x) + N\bar{u}(x)| > \vartheta$. First, we observe that the weak convergence $\bar{u}_h \circ g_h^{-1} \rightharpoonup \bar{u}$ in $H^1(\Gamma)$ implies the strong convergence in $C(\Gamma)$. On the other hand, from (5.24) with $s = 3/2$ we get

$$\|\bar{\varphi} - \bar{\varphi}_h\|_{C(\bar{\Omega}_h)} \le C_1 \|\bar{\varphi} - \bar{\varphi}_h\|_{H^{3/2}(\Omega_h)}$$

$$\le C_2 \left\{ \|\bar{u} - \bar{u}_h \circ g_h^{-1}\|_{L^2(\Gamma)} + h[\|\bar{u}\|_{H^{1/2}(\Gamma)} + 1] \right\} \to 0 \text{ with } h \to 0. \tag{6.7}$$

This inequality implies the strong convergence $\bar{\varphi}_h \circ g_h^{-1} \to \bar{\varphi}$ in $C(\Gamma)$. Therefore, there exists $h_1 > 0$ such that

$$\|(\bar{\varphi}_h \circ g_h^{-1} + \bar{u}_h \circ g_h^{-1}) - (\bar{\varphi} + N\bar{u})\|_{C(\Gamma)} < \frac{\vartheta}{2} \quad \text{for all } \ h \le h_1. \tag{6.8}$$

Thus, if $(\bar{\varphi} + N\bar{u})(x) > \vartheta$, then $(\bar{\varphi}_h \circ g_h^{-1} + \bar{u}_h \circ g_h^{-1})(x) > \vartheta/2$ for every $h \le h_1$. Using the identities (3.4) and (4.10), we have that $\bar{u}(x) = \bar{u}_h \circ g_h^{-1}(x) = \alpha$, therefore $(\bar{u}(x) - \bar{u}_h \circ g_h^{-1})(x) = 0$. Analogously we can prove that $(\bar{\varphi} + N\bar{u})(x) < -\vartheta$ leads to $\bar{u}(x) = \bar{u}_h \circ g_h^{-1}(x) = \beta$ and then $(\bar{u}(x) - \bar{u}_h \circ g_h^{-1})(x) = 0$ as well. This proves that $\bar{u}_h \circ g_h^{-1} - \bar{u} \in C_{\bar{u}}^{\vartheta}$ and hence (3.8) implies

$$J''(\bar{u})(\bar{u}_h \circ g_h^{-1} - \bar{u})^2 \ge \mu \|\bar{u}_h \circ g_h^{-1} - \bar{u}\|^2_{L^2(\Gamma)} \quad \text{for all } \ h \le h_2. \tag{6.9}$$

For the elements v_h in (6.6) we have that $v_h \to 0$ in $C(\Gamma)$ with $h \to 0$. On the other hand, the mapping J is of class C^2 in $L^2(\Gamma)$, therefore there exists $0 < h_2 \le h_1$ such that

$$\left|[J''(\bar{u}) - J''(v_h)](\bar{u}_h \circ g_h^{-1} - \bar{u})^2\right| \le \frac{\mu}{2} \|\bar{u}_h \circ g_h^{-1} - \bar{u}\|^2_{L^2(\Gamma)} \quad \text{for all } \ h \le h_2.$$

This inequality combined with (6.9) leads to

$$J''(v_h)(\bar{u}_h \circ g_h^{-1} - \bar{u})^2 \ge \frac{\mu}{2} \|\bar{u}_h \circ g_h^{-1} - \bar{u}\|^2_{L^2(\Gamma)}. \tag{6.10}$$

Upper Bounds for (6.6). Let us define $y, \varphi \in H^2(\Omega)$ as the solutions of the equations

$$\left\{ \begin{array}{rcll} -\Delta y + a(x, y) & = & 0 & \text{in } \Omega, \\ \partial_\nu y & = & \bar{u}_h \circ g_h^{-1} & \text{on } \Gamma, \end{array} \right. \tag{6.11}$$

and

$$\left\{\begin{array}{rcll} -\Delta\varphi + \dfrac{\partial a}{\partial y}(x,y)\varphi & = & \dfrac{\partial L}{\partial y}(x,y) & \text{in } \Omega, \\ \partial_\nu\varphi & = & 0 & \text{on } \Gamma. \end{array}\right. \tag{6.12}$$

Then we have

$$J'(\bar u_h \circ g_h^{-1})(\bar u - \bar u_h \circ g_h^{-1}) = \int_\Gamma (\varphi + N\bar u_h \circ_h^{-1})(\bar u - \bar u_h \circ g_h^{-1})d\sigma. \tag{6.13}$$

From (6.3) and (6.13) and taking into account (4.2), (4.3) and (5.9) we get

$$|J_h'(\bar u_h)(\bar u \circ g_h - \bar u_h) - J'(\bar u_h \circ g_h^{-1})(\bar u - \bar u_h \circ g_h^{-1})|$$

$$= \left| \int_{\Gamma_h} (\bar\varphi_h + N\bar u_h)(\bar u \circ g_h - \bar u_h)\, d\sigma_h - \int_\Gamma (\varphi + N\bar u_h \circ g_h^{-1})(\bar u - \bar u_h \circ g_h^{-1})d\sigma \right|$$

$$\le \int_{\Gamma_h} |(\bar\varphi_h + N\bar u_h) - (\varphi \circ g_h + N\bar u_h)||\bar u \circ g_h - \bar u_h| d\sigma_h$$

$$+Ch^2 \int_\Gamma |\varphi + N\bar u_h \circ g_h^{-1}||\bar u - \bar u_h \circ g_h^{-1}| d\sigma$$

$$\le \left\{ \|\bar\varphi_h - \varphi\|_{L^2(\Gamma_h)} + \|\varphi \circ g_h - \varphi\|_{L^2(\Gamma_h)} \right\} \|\bar u \circ g_h - \bar u_h\|_{L^2(\Gamma_h)}$$

$$+Ch^2 \|\varphi + N\bar u_h \circ g_h^{-1}\|_{L^2(\Gamma)} \|\bar u - \bar u_h \circ g_h^{-1}\|_{L^2(\Gamma)}$$

$$\le C(h^{5/3} + h^2)\|\bar u - \bar u_h \circ g_h^{-1}\|_{L^2(\Gamma)} \le Ch^{5/3}\|\bar u - \bar u_h \circ g_h^{-1}\|_{L^2(\Gamma)}, \tag{6.14}$$

the last estimate being a consequence of (5.35).

Finally (6.6), (6.10) and (6.14) lead to (6.1), which completes the proof. ∎

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